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# Post-Buckling Behavior of a Thick Circular Plate

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## Introduction

THE post-buckling behavior of circular plates has been the subject of a great deal of study for many years.[1,2] Wellford[3] and Rao[4] presented solutions for the post-buckling behavior of elastic circular plates by using a finite element method. The axisymmetric post-buckling behavior of plates has been recently tested experimentally.[5] All of the past investigators have been concerned mainly with in-plane compressive stresses and most of their studies were concerned with thin plate theory.

Using the average stress method, the authors have derived the nonlinear equations on the basis of von Kármán's assumption to study the large amplitude vibration[6] and post-buckling[7] problems for a thick rectangular plate in an arbitrary state of initial stress. In the present work, the previously derived nonlinear equations[8] of a transversely isotropic circular thick plate in a general state of nonuniform initial stress are used. The post-buckling problems of both simply supported and clamped axisymmetric circular plates subjected to uniform in-plane compression and a uniform bending stress acting along the edge are studied.

## Governing Equations

Consider a circular plate of uniform thickness $h$ and radius $a$ in a state of arbitrary edge loading. The state of applied stress is

$$\sigma_{rr} = \sigma_n + 2z\sigma_m / h \tag{1}$$

$\sigma_n$ and $\sigma_m$ are taken to be constants. It is comprised of a compressive plus a bending stress. The only nonzero stress resultants are

$$N_r = h\sigma_n, \quad M_r = h^2\sigma_m/6, \quad M_r^* = h^3\sigma_n/12 \tag{2}$$

The coordinate system was chosen such that the middle plane of the plate coincides with the $r$-$\theta$ plane. The origin of the coordinate system begins at the center of the plate with the positive $z$ axis upward.

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For an axisymmetric circular plate, the $\theta$ dependence can be dropped and the displacement field is simplified by

$$\xi_r(r,z,t) = u(r,t) + z\psi_r(r,t), \quad \xi_\theta = 0$$
$$\xi_z(r,z,t) = w(r,t) \tag{3}$$

Lateral loads and body forces are taken to be zero. For the static problem, the equations of motion are as follows.[8]

$$D(u_{,r}+\nu u/r)_{,r} + D(u_{,r}+\nu u/r)/r - D(u/r+\nu u_{,r})/r + D(1-\nu)w_{,r}^2/2r + Dw_{,r}w_{,rr} + (N_r u_{,r} + M_r\psi_{r,r})_{,r} + (N_r u_{,r} + M_r\psi_{r,r})/r = 0 \tag{4}$$

$$(N_r w_{,r})_{,r} + N_r w_{,r}/r + \kappa^2 G^* h(w_{,r}+\psi_r)_{,r} + \kappa^2 G^* h(w_{,r}+\psi_r)/r + D\epsilon_1(w_{,rr}+w_{,r}/r) + D\epsilon_{1,r}w_{,r} = 0 \tag{5}$$

$$(M_r u + M_r^*\psi_{r,r})_{,r} + (M_r u_{,r} + M_r^*\psi_{r,r})/r + D^*(\psi_{r,r}+\nu\psi_r/r)_{,r} + D^*(\psi_{r,r}+\nu\psi_r/r)/r - \kappa^2 G^* h(w_{,r}+\psi_r) - D^*(\psi_r/r+\nu\psi_{r,r})/r = 0 \tag{6}$$

where

$$\epsilon_1 = u_{,r} + w_{,r}^2/2$$

The boundary conditions are for the simply supported immovable plate

$$\bar{M}_r + \Delta M_r = M_r u_{,r} + M_r^*\psi_{r,r} + D^*(\psi_{r,r} + \nu\psi_r/r) = 0$$
$$w = u = 0 \quad \text{at} \quad r = a$$
$$u = \psi_r = w_{,r} = 0 \quad \text{at} \quad r = 0 \tag{7}$$

and for the clamped immovable plate

$$w = u = \psi_r = 0 \quad \text{at} \quad r = a$$
$$u = \psi_r = w_{,r} = 0 \quad \text{at} \quad r = 0 \tag{8}$$

Displacements of the following form satisfy the geometric boundary conditions:

$$u(r,t) = u(A_1 y + A_2 y^3 + A_3 y^5 + A_4 y^7)$$
$$w(r,t) = w(1 + B_1 y^2 + B_2 y^4)$$
$$\psi_r(r,t) = \Psi(C_1 y + C_2 y^3) \tag{9}$$

where

$$y = r/a, \quad A_1 = (5-3\nu)/6, \quad A_2 = -(3-\nu)$$
$$A_3 = 2(5-\nu)/3, \quad A_4 = -(7-\nu)/6$$

For the simply supported plate

$$B_1 = -(6+2\nu)/(5+\nu), \qquad B_2 = (1+\nu)/(5+\nu)$$
$$C_1 = 2(6+2\nu)/(5+\nu), \qquad C_2 = -4(1+\nu)/(5+\nu)$$

and for the clamped plate

$$B_1 = -2, \quad B_2 = 1, \quad C_1 = 4, \quad C_2 = -4$$

Equations (10-12) are obtained by substituting the assumed displacement field of Eq. (9) into the equations of motion, Eqs. (4-6), and solving by Galerkin's method.

$$C_{11}U + C_{13}\Psi + C_{14}W^2 = 0 \tag{10}$$

$$C_{21}W + C_{22}\Psi + C_{23}UW + C_{24}W^3 = 0 \tag{11}$$

$$C_{31}U + C_{32}W + C_{33}\Psi = 0 \tag{12}$$

The coefficients are the same as Ref. 8 and the nondimensional parameters are

$$U = u/h, \quad W = w/h, \quad S = G^*/G, \quad K = N_r a^2/D^*$$

$$\beta = \sigma_m/\sigma_n, \quad \kappa^2 = \pi^2/12, \quad D^* = Eh^3/12(1-\nu^2)$$

$$D = Eh/(1-\nu^2), \quad Gh^3/12 = D^*(1-\nu)/2$$

## Results and Discussions

Many parameters affect the post-buckling behavior. To verify the accuracy of the present results, the buckling coefficient with no bending stress is the first to be considered, and the result is compared with Pardoen's[9] result. In Table 1 it can be seen that the $K_{cr}$ value that is calculated by the present formulation coincides with Pardoen's result very well for the simply supported circular plate and is a slightly higher value for the clamped circular plate.

Plots are made of $K$ vs $W$ in Fig. 1. The values for $a/h$, $S$, $\beta$, and $\nu$ are chosen to be 5, 1, 0, 0.3 for a thick plate and 100, 1, 0, 0.3 for a thin plate, respectively. The effect of shear deformation can be observed from the differences of the deflections between the thick plates and thin plates, where the deflections of the thick plate are larger than the thin plate. Figure 2 shows the effect of the transversely isotropic coefficient; $S$, $a/h$, $\beta$, and $\nu$ are equal to 5, 0, 0.3, respectively. It is seen that the larger the transversely isotropic coefficient $S$ is, the lower the deflection is. It means that the post-buckled strength will reduce when the transverse shear resistance is large. The effect of Poisson's ratio $\nu$ is shown in Fig. 3, where the $a/h$, $S$, and $\beta$ are equal to 5, 1, 0, respectively. The deflection increases with the decreasing Poisson's ratio for both simply supported and clamped plates.

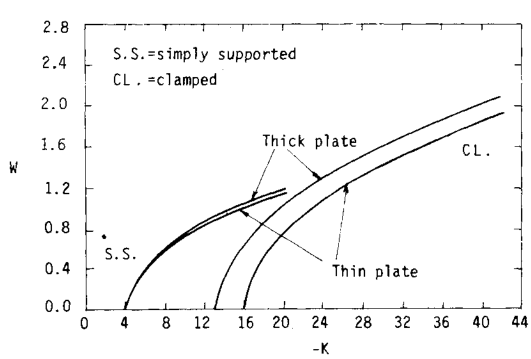



**Fig. 1 Relation between the post-buckling coefficient and the central deflection of thick and thin plates. Thick plate: $a/h = 5.0$, $S = 1.0$, $\nu = 0.3$, $\beta = 0.0$; thin plate: $a/h = 100$, $S = 1.0$, $\nu = 0.3$, $\beta = 0.0$.**

**Table 1 Values of buckling load $K_{cr}$ with no bending stress for clamped (CL) and simply supported (SS) plates**

| $a/h$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| CL | 13.1234 | 15.1533 | 15.7783 | 15.9641 | 15.9910 |
| SS | 3.9897 | 4.1505 | 4.1929 | 4.2050 | 4.2061 |

Pardoen's results: $K_{cr} = 14.6825$ for clamped circular plates
$K_{cr} = 4.1978$ for simply supported circular plates

Plots of $W$ vs $K/K_{cr}$ for various values of $\beta$ are made in Fig. 4, where $a/h$, $S$, and $\nu$ are equal to 5, 1, and 0.3, respectively. The bending stress effect is shown to increase the deflection when $\beta$ is positive. For comparison of the effect of bending stress on the post-buckling behavior of thick plates and thin plates, $K/K_{cr}$ is chosen to be 2.0, and plots of $W$ vs $\beta$ are made in Fig. 5. We can see that the bending stress coefficient $\beta$ has little effect on thin plates, but has a significant effect on thick plates. Also, clamped edges show thick plate effects more than simply supported edges do. This is a general feature of thick plates as pointed out by Brunelle.[10]

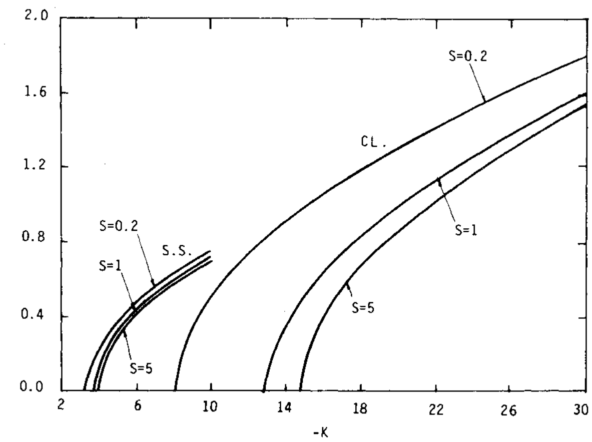



**Fig. 2 Central deflection vs post-buckling load for various values of transversely isotropic parameters; $a/h = 5.0$, $\nu = 0.3$, $\beta = 0.0$.**

**Fig. 3 Central deflection vs post-buckling load for various values of Poisson's ratio; $a/h = 5.0$, $S = 1.0$, $\beta = 0.0$.**

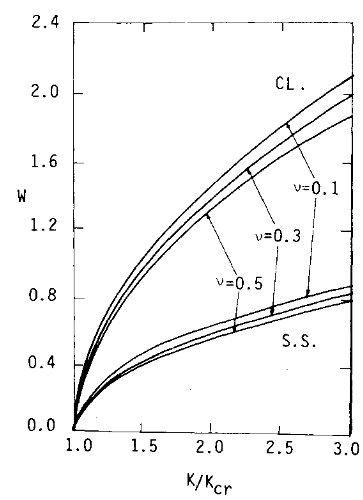
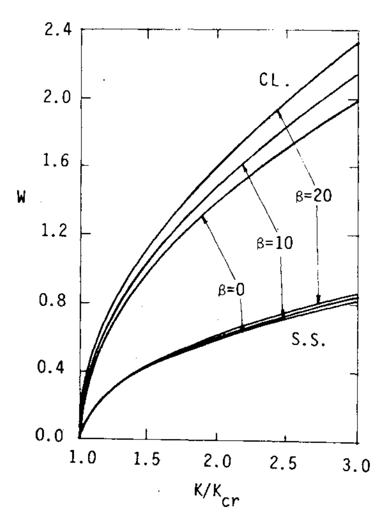



**Fig. 4 Central deflection vs post-buckling load for various values of $\beta$; $a/h = 5.0$, $S = 1.0$, $\nu = 0.3$.**



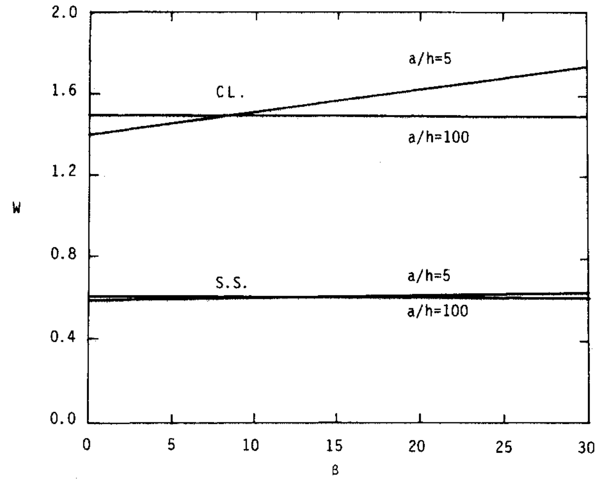



**Fig. 5 Central deflection vs $\beta$ for thick and thin plates; $K/K_{cr}=2.0$, $S=1.0$, $\nu=0.3$.**

## Conclusions

The preliminary results indicate the following.

1) By using the present method one can obtain the buckling coefficient to match Pardoen's results.

2) The thick plate buckles at a lower $-K$ than the thin plate does, and the thick plate post-buckled deflection is larger than the corresponding thin plate.

3) Deflections increase with the decreasing of the transversely isotropic coefficient $S$.

4) The post-buckled deflections of the plate increase when the Poisson's ratio decreases.

5) The deflection increases with increasing bending stress coefficient. The effects are significant for thick plates.

6) Thick plate effects are accentuated by increasing the boundary restraint.

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# Linear System Identification via Poisson Moment Functionals

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## Introduction

THE method of Poisson moment functionals (PMF) provides an effective system identification algorithm in the time domain for linear time-invariant systems. First introduced by Fairman and Shen[1,2] and Fairman,[3] the method was systematically adapted to discrete linear time-invariant and time-varying systems in a series of papers by Saha and Prasada Rao[4-8] and Sivakumar and Prasada Rao.[9] In that series of papers, they provide algorithms for the synthesis of transfer functions from time domain data, both for homogeneous initial conditions and, by an augmentation procedure, for arbitrary unknown initial conditions. More importantly, they provide an algorithm for the direct identification of the elements of the system state matrix. However, that method has been developed only for the force-free case, wherein the initial conditions are fully known.

The desirable attributes of identification via PMF are now well documented. Among them are the following.

1) Identification is done in continuous time, despite the fact that process signals are sampled in discrete time.

2) The method is somewhat naturally immune to zero-mean additive noise.

3) The PMF's of the process signals can be obtained online as the response of a Poisson filter chain.

Application of the method to the state identification of structures has been hampered by limitation of existing algorithms to the force-free case. This is nearly always too restrictive, for several reasons. First, the most accurate dynamic test methods involve one or more actuators exciting the structure; and second, the importance of substructure input matrices becomes equal to that of the state matrices when a large structure is synthesized from identified substructure parameters (see, for example, Hale and Bergman[10]).

The purpose of this Note, then, is to provide an extension of the algorithm, first proposed in Ref. 4, to the synthesis of state equations when the system to be identified is subjected to forces, internal and/or external.

## Poisson Moment Functionals

The PMF transform takes a signal $f(t)$ over the interval $(0,t_0)$ and converts it into a set of real numbers

$$M_k[f(t)]_{t_0}=f_k=\int_0^{t_0} f(t)p_k(t_0-t)\,dt, \quad (k=0,1,2,\ldots) \qquad (1)$$

where

$$p_k(t)=t^k\exp(-\lambda t)/k! \qquad (2)$$

$$\lambda(\text{real})\geq 0, \text{ and } f_{-1}=f(t_0) \qquad (3)$$

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